On \textit{gso}-Closed Sets in Topological Spaces

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Authors’ contributions

Author PE designed the study, wrote the protocol. Author MII managed the literature searches, did the theoretical work and wrote the first draft of the manuscript. Both authors read and approved the final manuscript.

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Abstract

In this paper, a new kind of closed sets called generalized semi open-closed sets (briefly \textit{gso}-closed sets) are introduced in topological spaces. A subset $A$ of a topological space $X$ is called a \textit{gso}-closed set if $A$ is both a \textit{g}-closed set and a semi-open set in $X$. The properties of the \textit{gso}-closed sets are investigated and they are compared with the existing relevant generalized closed sets. The generalized semi-open continuous function between topological spaces are also defined and their properties are investigated.

Keywords: Generalized closed sets; continuous functions; generalized continuous functions.

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1 Introduction

The concept of closed sets in topological spaces have many important properties such as closed subset of a compact space is compact; closed subset of a normal space is normal; closed subset of a complete uniform space is complete, etc. The generalized closed sets, simply \textit{g}-closed sets, which are weaker form of the closed sets were introduced and studied by N. Levine [1] in a topological space in 1970. He defined a set $A$ to be a \textit{g}-closed set in a topological space if its closure is contained in every open super set of $A$.
Since then many mathematicians introduce and investigate different kind of generalized closed sets in topological spaces. Some recent works on generalized closed sets can be found in [2], [3], [4], [5] and [6]. Continuous functions are one of the core concepts of topological spaces defined in terms of open sets as well as closed sets. With the introduction of different kind of generalized closed sets, there are many different kind of continuous functions defined in the literature. In this paper, we introduce a new kind of closed set called the generalized semi open-closed set (briefly gso-closed set) in topological spaces. A subset $A$ of a topological space $(X, \tau)$ is called a $gso$-closed set if $A$ is both a $g$-closed set and a semi-open set in the $(X, \tau)$. We study its properties and compare this with the existing relevant generalized closed sets. In the second part of this paper, we define the generalized semi-open continuous functions between topological spaces as an application of these $gso$-closed sets and investigate their properties.

1.2 Generalized Continuous Functions

Continuity of functions between topological spaces is one of the core concepts of topological spaces which is defined in terms of open sets as well as closed sets. Here we recall some kind of generalized continuous function defined by the relevant generalized closed sets.

**Definition 1.3.** Let $X$ and $Y$ be two topological spaces. A function $f : X \to Y$ is called a

(i) $g$-continuous [12] if $f^{-1}(V)$ is a $g$-closed set of $X$ for every closed set $V$ of $Y$.

(ii) $gs$-continuous [13] if $f^{-1}(V)$ is a $gs$-closed set of $X$ for every closed set $V$ of $Y$.

(iii) $wg$-continuous [11] if $f^{-1}(V)$ is a $wg$-closed set of $X$ for every closed set $V$ of $Y$.

(iv) $rwg$-continuous [12] if $f^{-1}(V)$ is a $rwg$-closed set of $X$ for every closed set $V$ of $Y$.

2 Results and Discussion

**Definition 2.1.** A subset $A$ of a topological space $X$ is called a generalized semi open-closed set (briefly $gso$-closed set) if $A$ is both a $g$-closed set and a semi-open set in $X$.

It is clear that every $gso$-closed set is a $g$-closed set in a topological space.
Example 2.1. Consider a linearly ordered topological space, defined in Faber, [14]. A basic example is the real numbers with the standard order. Or more generally, a preference-ordered topological space, defined in Alcantud, [15]. Then the sets with the form

\[ \{ x \in X : a \leq x \leq b \} \]

are gso-closed sets when \( a < b \).

Theorem 2.2. The union of two gso-closed sets in a topological space \( X \) is also a gso-closed set in \( X \).

Proof. Let \( A \) and \( B \) be two gso-closed sets in a topological space \( X \). Then \( A \) and \( B \) are both g-closed sets and semi-open sets in \( X \). Therefore, \( A \cup B \) is both g-closed set and semi-open set and hence it is a \( gso \)-closed set in \( X \).

The intersection of two \( gso \)-closed sets in a topological space \( X \) is generally not a \( gso \)-closed set in \( X \). We see this by the following example. Let \( X = \{ a, b, c \} \) be with topology \( \tau = \{ X, \emptyset, \{ a \} \} \). Then, \( A = \{ a, b \} \) and \( B = \{ a, c \} \) are \( gso \)-closed sets. But their intersection, \( A \cap B = \{ a \} \) is not a \( gso \)-closed set in \( X \).

Theorem 2.3. If \( A \) and \( B \) are both closed and open sets in a topological space \( X \), then \( A \cap B \) is a \( gso \)-closed set in \( X \).

Proof. Suppose that \( A \) and \( B \) are both open and closed sets in a topological space \( X \). Then, \( A \cap B \) is a closed set and so is a g-closed set. Also, \( A \cap B \) is an open set and so is a semi-open set in \( X \). Hence \( A \cap B \) is a \( gso \)-closed set in \( X \).

Theorem 2.4. If \( A \) is a \( gso \)-closed set in a topological space \( X \) and \( B \) is an open set in \( X \) such that \( B \subseteq A \), then \( B \) is a \( gso \)-closed set in \( X \).

Proof. Suppose that \( A \) is a \( gso \)-closed set in a topological space \( X \). Then, \( cl(A) \subseteq U \) whenever \( A \subseteq U \). \( U \) is open in \( X \) and \( A \subseteq cl(int(A)) \). We now suppose that \( B \subseteq A \) and \( B \) is open in \( X \) and \( A \subseteq U \). \( U \) is open in \( X \). Then, we get \( cl(B) \subseteq U \) whenever \( B \subseteq U \), \( U \) is open in \( X \). Thus, \( B \) is a g-closed set in \( X \). As \( cl(B) \subseteq cl(int(B)) \), we also get \( B \subseteq cl(int(B)) \). Thus, \( B \) is a semi-open set in \( X \). Hence \( B \) is a \( gso \)-closed set in \( X \).

Theorem 2.5. If \( A \) is both closed and open set in a topological space \( X \), then \( A \) is a \( gso \)-closed set in \( X \).

Proof. Suppose that \( A \) is both open and closed set in a topological space \( X \). Then, \( A \) is a \( g \)-closed set and a semi-open set. Hence \( A \) is a \( gso \)-closed set in \( X \).

Theorem 2.6. If a \( gso \)-closed set \( A \) is a pre-closed set in a topological space \( X \), then \( A \) is a regular closed set in \( X \).

Proof. Suppose that the \( gso \)-closed set \( A \) is a pre-closed set in a topological space \( X \). Then, we get \( A \subseteq cl(int(A)) \) and \( cl(int(A)) \subseteq A \). Thus \( A = cl(int(A)) \). Hence \( A \) is a regular closed set in \( X \).

Theorem 2.7. Every \( gso \)-closed set in a topological space \( X \) is a \( wg \)-closed set in \( X \).

Proof. Let \( A \) be a \( gso \)-closed set in a topological space \( X \). Then, \( cl(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is open in \( X \) and \( A \subseteq cl(int(A)) \). Let \( A \subseteq U \), \( U \) is open in \( X \). Then, \( cl(int(A)) \subseteq cl(A) \subseteq U \). Hence \( A \) is a \( wg \)-closed set in \( X \).

The converse of the above theorem need not be true as seen in the following example. Let \( X = \{ a, b, c, d \} \) be with topology \( \tau = \{ X, \emptyset, \{ a \}, \{ b \}, \{ a, b \}, \{ a, b, c \} \} \). Then, the set \( A = \{ d \} \) is a \( wg \)-closed set in \( X \) but not a \( gso \)-closed set in \( X \).

Theorem 2.8. Every \( gso \)-closed set in a topological space \( X \) is a \( gs \)-closed set in \( X \).

Proof. Let \( A \) be a \( gso \)-closed set in a topological space \( X \). Then, \( cl(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is open in \( X \) and \( A \subseteq cl(int(A)) \). Let \( A \subseteq U \), \( U \) is open in \( X \). Since \( scl(A) \subseteq cl(A) \), we get \( scl(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is open set in \( X \). Hence \( A \) is a \( gs \)-closed set in \( X \).

The converse of the above theorem need not be true as seen in the following example. Let \( X = \{ a, b, c, d \} \) be with topology \( \tau = \{ X, \emptyset, \{ a \}, \{ b \}, \{ a, b \}, \{ a, b, c \} \} \). Then, the set \( A = \{ a \} \) is a \( gs \)-closed set in \( X \) but not a \( gso \)-closed set in \( X \).

Theorem 2.9. Every \( gso \)-closed set in a topological space \( X \) is a \( rwg \)-closed set in \( X \).
Let $A$ be a gso-closed set in a topological space $X$. Then, $cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $X$ and $A \subseteq cl(int(A))$. Let $A \subseteq U$, $U$ is open in $X$. Then, $cl(int(A)) \subseteq U$ whenever $A \subseteq U$ and $U$ is regular open in $X$. Hence $A$ is a rgw-closed set in $X$.

**Definition 2.2.** Let $X$ and $Y$ be two topological spaces. A function $f : X \to Y$ is called a generalized semi-open-continuous function (briefly gso-continuous function) if $f^{-1}(V)$ is a gso-closed set of $X$ for every closed set $V$ of $Y$.

**Example 2.10.** Let $X = \{a, b\}$ be with topology $\tau = \{X, \emptyset, \{a\}\}$ and let $Y = \{a, b, c\}$ be with topology $\sigma = \{Y, \emptyset, \{a, b\}\}$. Define $f : X \to Y$ by $f(a) = a$ and $f(b) = c$. Then, $f$ is a gso-continuous function.

In general, the composition of two gso-continuous functions need not be a gso-continuous function. This can be seen in the following example. Let $X = \{a, b\}$ be with topology $\tau = \{X, \emptyset, \{a\}\}$ and let $Y = \{a, b, c\}$ be with topology $\sigma = \{Y, \emptyset, \{a, b\}\}$ and $Z = \{a, b, c\}$ be with topology $\eta = \{Z, \emptyset, \{a, c\}\}$. Define $f : X \to Y$ by $f(a) = a$ and $f(b) = c$, and $g : Y \to Z$ by $g(a) = a$ and $g(c) = b$. Then, $f$ and $g$ are gso-continuous functions but the composition $g \circ f$ is not a gso-continuous function.

**Theorem 2.11.** Let $X$, $Y$ and $Z$ be two topological spaces. Let $f : X \to Y$ be a gso-continuous function and $g : Y \to Z$ be a continuous function. Then, $g \circ f$ is a gso-continuous function.

**Proof.** Suppose that $A$ be a closed set in $Z$. Then, $g^{-1}(A)$ be a closed set in $Y$ and so $f^{-1}(g^{-1}(A))$ is a gso-closed set in $X$. That is, $(g \circ f)^{-1}(A)$ is a gso-closed set in $X$. Hence $g \circ f$ is a gso-continuous function.

**Theorem 2.12.** Let $X$ and $Y$ be two topological spaces. Let $f : X \to X$ be an identity function and $g : X \to Y$ be a gso-continuous function. Then, $g \circ f$ is a gso-continuous function.

**Proof.** Suppose that $A$ be a closed set in $Y$. Then, $g^{-1}(A)$ is a gso-closed set in $X$ and then, $f^{-1}(g^{-1}(A)) = g^{-1}(A)$ is a gso-closed set in $X$. That is, $(g \circ f)^{-1}(A)$ is a gso-closed set in $X$. Hence $g \circ f$ is a gso-continuous function.

**Theorem 2.13.** Every gso-continuous function is a g-continuous function.

**Proof.** Suppose that $f : X \to Y$ is a gso-continuous function and let $A$ be a closed set in $Y$. Then, $f^{-1}(A)$ is a gso-closed set in $X$. Then by the definition of gso-closed set, $f^{-1}(A)$ is a g-closed set in $X$. Hence $f$ is a g-continuous function.

**Theorem 2.14.** Every gso-continuous function is a wg-continuous function.

**Proof.** Suppose that $f : X \to Y$ is a gso-continuous function and let $A$ be a closed set in $Y$. Then, $f^{-1}(A)$ is a gso-closed set in $X$. Then by Theorem 2.7, $f^{-1}(A)$ is a wg-closed set in $X$. Hence $f$ is a wg-continuous function.

**Theorem 2.15.** Every gso-continuous function is a g ws-continuous function.

**Proof.** Suppose that $f : X \to Y$ is a gso-continuous function and let $A$ be a closed set in $Y$. Then, $f^{-1}(A)$ is a gso-closed set in $X$. Then by Theorem 2.8, $f^{-1}(A)$ is a g ws-closed set in $X$. Hence $f$ is a g ws-continuous function.

**Theorem 2.16.** Every gso-continuous function is a rgw-continuous function.

**Proof.** Suppose that $f : X \to Y$ is a gso-continuous function and let $A$ be a closed set in $Y$. Then, $f^{-1}(A)$ is a gso-closed set in $X$. Then by Theorem 2.9, $f^{-1}(A)$ is a rgw-closed set in $X$. Hence $f$ is a rgw-continuous function.

**3 Conclusions.**

In this paper, we defined a new kind of generalized closed set, called generalized semi open-closed sets (briefly gso-closed sets), in a topological space $X$. A subset $A$ of $X$ is called a gso-closed set if $A$ is both a g-closed set and semi-open set in $X$. This gso-closed set became some of the relevant generalized closed sets but not conversely. The generalized semi-open continuous function (briefly gso-continuous function) between two topological spaces are also defined as an application of the gso-closed set. From these gso-continuous functions some of the relevant generalized continuous functions were derived.
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Competing Interests

Authors have declared that no competing interests exist.

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