Corrigendum: On Summing Formulas for Generalized Fibonacci and Gaussian Generalized Fibonacci Numbers

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Author’s contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

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ABSTRACT

In this paper, closed forms of the summation formulas for generalized Fibonacci and Gaussian generalized Fibonacci numbers are presented. Then, some previous results are recovered as particular cases of the present results. As special cases, we give summation formulas of Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal, Jacobsthal-Lucas numbers and Gaussian Fibonacci, Gaussian Lucas, Gaussian Pell, Gaussian Pell-Lucas, Gaussian Jacobsthal, Gaussian Jacobsthal-Lucas numbers.

Keywords: Fibonacci numbers; gaussian fibonacci numbers; lucas numbers; pell numbers; jacobsthal numbers; sum formulas; summing formulas.

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1 INTRODUCTION

In 1965, Horadam [1] defined a generalization of Fibonacci sequence, that is, he defined a second-order linear recurrence sequence \( \{ W_n(W_0, W_1; r, s) \} \), or simply \( \{ W_n \} \), as follows:

\[
W_n = r W_{n-1} + s W_{n-2}; \quad W_0 = a, \quad W_1 = b, \quad (n \geq 2)
\]

where \( W_0, W_1 \) are arbitrary complex numbers and \( r, s \) are real numbers, see also Horadam [2], [3] and [4]. Now these generalized Fibonacci numbers \( \{ W_n(a, b; r, s) \} \) are also called Horadam numbers.

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The sequence \( \{W_n\}_{n \geq 0} \) can be extended to negative subscripts by defining
\[
W_{-n} = -\frac{r}{s}W_{-(n-1)} + \frac{1}{s}W_{-(n-2)}
\]
for \( n = 1, 2, 3, \ldots \) when \( s \neq 0 \). Therefore, recurrence (1.1) holds for all integer \( n \).

For some specific values of \( a, b, r, s \) and \( s \), it is worth presenting these special Horadam numbers in a table as a specific name. In literature, for example, the following names and notations (see Table 1) are used for the special cases of \( r, s \) and initial values.

### Table 1. A few special case of generalized Fibonacci sequences.

<table>
<thead>
<tr>
<th>Name of sequence</th>
<th>Notation: ( W_n(a, b; r, s) )</th>
<th>No in oeis.org: [5]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fibonacci</td>
<td>( F_n = W_n(0; 1; 1, 1) )</td>
<td>A000045</td>
</tr>
<tr>
<td>Lucas</td>
<td>( L_n = W_n(2; 1; 1, 1) )</td>
<td>A000032</td>
</tr>
<tr>
<td>Pell</td>
<td>( P_n = W_n(0; 1; 2, 1) )</td>
<td>A000129</td>
</tr>
<tr>
<td>Pell-Lucas</td>
<td>( Q_n = W_n(2; 2; 2, 1) )</td>
<td>A002203</td>
</tr>
<tr>
<td>Jacobsthal</td>
<td>( J_n = W_n(0; 1; 1, 2) )</td>
<td>A001045</td>
</tr>
<tr>
<td>Jacobsthal-Lucas</td>
<td>( j_n = W_n(2; 1; 1, 2) )</td>
<td>A014551</td>
</tr>
</tbody>
</table>

A Gaussian integer \( z \) is a complex number whose real and imaginary parts are both integers, i.e., \( z = a + ib, a, b \in \mathbb{Z} \). If we use together sequences of integers defined recursively and Gaussian type integers, we obtain a new sequences of complex numbers such as Gaussian Fibonacci, Gaussian Lucas, Gaussian Pell, Gaussian Pell-Lucas and Gaussian Jacobsthal numbers; Gaussian Padovan and Gaussian Pell-Padovan numbers; Gaussian Tribonacci numbers.

Gaussian generalized Fibonacci (Horadam) numbers \( \{GW_n\}_{n \geq 0} = \{GW_n(GW_0, GW_1; r, s)\}_{n \geq 0} \) are defined by
\[
GW_n = rGW_{n-1} + sGW_{n-2} \tag{1.2}
\]
with the initial conditions
\[
GW_0 = W_0 + (\frac{r}{s}GW_0 + \frac{1}{s}GW_1)i, \quad GW_1 = W_1 + W_0i
\]
not all being zero. The sequences \( \{GW_n\}_{n \geq 0} \) can be extended to negative subscripts by defining
\[
GW_{-n} = -\frac{r}{s}GW_{-(n-1)} + \frac{1}{s}GW_{-(n-2)} = -\frac{r}{s}GW_{n+1} + \frac{1}{s}GW_{n+2}
\]
for \( n = 1, 2, 3, \ldots \). Therefore, recurrence (1.2) holds for all integer \( n \). Note that for \( n \geq 0 \)
\[
GW_n = W_n + iW_{n-1}
\]
and
\[
GW_{-n} = W_{-n} + iW_{-n-1}
\]
For some specific values of \( W_0, W_1, r, s \), it is worth presenting these special Gaussian Horadam numbers in a table as a specific name. In literature, for example, the following names and notations (see Table 2) are used for the special cases of \( r, s \) and initial values.
### Table 2. A few special case of generalized Gaussian Fibonacci sequences

<table>
<thead>
<tr>
<th>Name of sequence</th>
<th>Notation: $GW_n(GW_0, GW_1; r, s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gaussian Fibonacci</td>
<td>$GF_n = GW_n(1; 1, 1, 1)$</td>
</tr>
<tr>
<td>Gaussian Lucas</td>
<td>$GL_n = GW_n(2 - i; 1 + 2i; 1, 1)$</td>
</tr>
<tr>
<td>Gaussian Pell</td>
<td>$GP_n = GW_n(1; 2; 1)$</td>
</tr>
<tr>
<td>Gaussian Pell-Lucas</td>
<td>$GQ_n = GW_n(2 - 2i; 2 + 2i; 2, 1)$</td>
</tr>
<tr>
<td>Gaussian Jacobsthal</td>
<td>$GJ_n = GW_n(\frac{1}{2}; 1; 1, 2)$</td>
</tr>
<tr>
<td>Gaussian Jacobsthal-Lucas</td>
<td>$GJ_n = GW_n(2 - \frac{1}{2}i; 1 + 2i; 1, 2)$</td>
</tr>
</tbody>
</table>

In this work, we investigate summation formulas of generalized Fibonacci and Gaussian generalized Fibonacci numbers. Some summing formulas of the Pell and Pell-Lucas numbers are well known and given in [6], [7], see also [8]. For linear sums of Fibonacci, Tribonacci, Tetranacci, Pentanacci and Hexanacci numbers, see [9], [10], [11], [12], [13], and [14] respectively.

## 2 SUMMING FORMULAS OF GENERALIZED FIBONACCI NUMBERS WITH POSITIVE SUBSCRIPTS

The following Theorem presents some linear summing formulas of generalized Fibonacci numbers with positive subscripts.

**Theorem 2.1.** For $n \geq 0$ we have the following formulas:

(a) *(Sum of the generalized Fibonacci numbers)* If $r + s - 1 \neq 0$, then

\[
\sum_{k=0}^{n} W_k = \frac{W_{n+2} + (1-r)W_{n+1} - W_1 + (r-1)W_0}{r + s - 1}.
\]

(b) If $(r - s + 1)(r + s - 1) \neq 0$ then

\[
\sum_{k=0}^{n} W_{2k} = \frac{(1-s)W_{2n+2} + rSW_{2n+1} + (s-1)W_2 - rSW_1 + (r^2 - s^2 + 2s - 1)W_0}{(r - s + 1)(r + s - 1)}
\]

and

\[
\sum_{k=0}^{n} W_{2k+1} = \frac{rW_{2n+2} + (s-s^2)W_{2n+1} - rW_2 + (r^2 + s - 1)W_1}{(r - s + 1)(r + s - 1)}.
\]

(c) If $r \neq 0 \land s = 1$ then

\[
\sum_{k=0}^{n} W_{2k} = \frac{W_{2n+1} - W_1 + rW_0}{r}
\]

and

\[
\sum_{k=0}^{n} W_{2k+1} = \frac{W_{2n+2} - W_2 + rW_1}{r}.
\]

Note that (c) is a special case of (b).

**Proof.**
(a) Using the recurrence relation

\[ W_n = rW_{n-1} + sW_{n-2} \]

i.e.

\[ sW_{n-2} = W_n - rW_{n-1} \]

we obtain

\[
\begin{align*}
  sW_0 &= W_2 - rW_1 \\
  sW_1 &= W_3 - rW_2 \\
  sW_2 &= W_4 - rW_3 \\
  & \vdots \\
  sW_{n-2} &= W_n - rW_{n-1} \\
  sW_{n-1} &= W_{n+1} - rW_n \\
  sW_n &= W_{n+2} - rW_{n+1}.
\end{align*}
\]

If we add the above equations by side by, we get

\[ \sum_{k=0}^{n} W_k = \frac{W_{n+2} + (1 - r)W_{n+1} - W_1 + (r - 1)W_0}{r + s - 1}. \]  \hspace{1cm} (2.1)

(b) and (c) Using the recurrence relation

\[ W_n = rW_{n-1} + sW_{n-2} \]

i.e.

\[ rW_{n-1} = W_n - sW_{n-2} \]

we obtain

\[
\begin{align*}
  rW_3 &= W_4 - sW_2 \\
  rW_5 &= W_6 - sW_4 \\
  rW_7 &= W_8 - sW_6 \\
  & \vdots \\
  rW_{2n+1} &= W_{2n+2} - sW_{2n} \\
  rW_{2n+3} &= W_{2n+4} - sW_{2n+2}
\end{align*}
\]

Now, if we add the above equations by side by, we get

\[ r(-W_1 + \sum_{k=0}^{n} W_{2k+1}) = (W_{2n+2} - W_2 - W_0 + \sum_{k=0}^{n} W_{2k}) - s(-W_0 + \sum_{k=0}^{n} W_{2k}). \]  \hspace{1cm} (2.1)

Similarly, using the recurrence relation

\[ W_n = rW_{n-1} + sW_{n-2} \]

i.e.

\[ rW_{n-1} = W_n - sW_{n-2} \]
we write the following obvious equations:

\[ rW_2 = W_3 - sW_1 \]
\[ rW_4 = W_5 - sW_3 \]
\[ rW_6 = W_7 - sW_5 \]
\[ rW_8 = W_9 - sW_7 \]
\[ \vdots \]
\[ rW_{2n} = W_{2n+1} - sW_{2n-1} \]
\[ rW_{2n+2} = W_{2n+3} - sW_{2n+1}. \]

Now, if we add the above equations by side by, we obtain

\[ r(-W_0 + \sum_{k=0}^{n} W_{2k}) = (-W_1 + \sum_{k=0}^{n} W_{2k+1}) - s(-W_{2n+1} + \sum_{k=0}^{n} W_{2k+1})). \] (2.2)

Then, solving the system (2.1)-(2.2), the required results of (b) and (c) follow.

Taking \( r = s = 1 \) in Theorem 2.1 (a) and (b), we obtain the following Proposition.

**Proposition 2.1.** If \( r = s = 1 \) then for \( n \geq 0 \) we have the following formulas:

(a) \( \sum_{k=0}^{n} W_k = W_{n+2} - W_1. \)

(b) \( \sum_{k=0}^{n} W_{2k} = W_{2n+1} - W_1 + W_0. \)

(c) \( \sum_{k=0}^{n} W_{2k+1} = W_{2n+2} - W_2 + W_1. \)

From the above Proposition, we have the following Corollary which gives linear sum formulas of Fibonacci numbers (take \( W_n = F_n \) with \( F_0 = 0, F_1 = 1 \)).

**Corollary 2.2.** For \( n \geq 0 \), Fibonacci numbers have the following properties:

(a) \( \sum_{k=0}^{n} F_k = F_{n+2} - 1. \)

(b) \( \sum_{k=0}^{n} F_{2k} = F_{2n+1} - 1. \)

(c) \( \sum_{k=0}^{n} F_{2k+1} = F_{2n+2}. \)

Taking \( W_n = L_n \) with \( L_0 = 2, L_1 = 1 \) in the last Proposition, we have the following Corollary which presents linear sum formulas of Lucas numbers.

**Corollary 2.3.** For \( n \geq 0 \), Lucas numbers have the following properties:

(a) \( \sum_{k=0}^{n} L_k = L_{n+2} - 1. \)

(b) \( \sum_{k=0}^{n} L_{2k} = L_{2n+1} + 1. \)

(c) \( \sum_{k=0}^{n} L_{2k+1} = L_{2n+2} - 2. \)

Taking \( r = 2, s = t = 1 \) in Theorem 2.1 (a) and (b), we obtain the following Proposition.

**Proposition 2.2.** If \( r = 2, s = t = 1 \) then for \( n \geq 0 \) we have the following formulas:

(a) \( \sum_{k=0}^{n} W_k = \frac{1}{2}(W_{n+2} - W_{n+1} - W_1 + W_0). \)

(b) \( \sum_{k=0}^{n} W_{2k} = \frac{1}{2}(W_{2n+1} - W_1 + 2W_0). \)

(c) \( \sum_{k=0}^{n} W_{2k+1} = \frac{1}{2}(W_{2n+2} - W_2 + 2W_1). \)
From the last Proposition, we have the following Corollary which gives linear sum formulas of Pell numbers (take $W_n = P_n$ with $P_0 = 0, P_1 = 1$).

**Corollary 2.4.** For $n \geq 0$, Pell numbers have the following properties:

(a) $\sum_{k=0}^{n} P_k = \frac{1}{2}(P_{n+2} - P_{n+1} - 1)$.
(b) $\sum_{k=0}^{n} P_{2k} = \frac{1}{2}(P_{2n+1} - 1)$.
(c) $\sum_{k=0}^{n} P_{2k+1} = \frac{1}{2}P_{2n+2}$.

Taking $W_n = Q_n$ with $Q_0 = 2, Q_1 = 2$ in the last Proposition, we have the following Corollary which presents linear sum formulas of Pell-Lucas numbers.

**Corollary 2.5.** For $n \geq 0$, Pell-Lucas numbers have the following properties:

(a) $\sum_{k=0}^{n} Q_k = \frac{1}{2}(Q_{n+2} - Q_{n+1})$.
(b) $\sum_{k=0}^{n} Q_{2k} = \frac{1}{4}Q_{2n+1} + 2$.
(c) $\sum_{k=0}^{n} Q_{2k+1} = \frac{1}{2}(Q_{2n+2} - 2)$.

If $r = 1, s = 2$ then $(r - s + 1)(r + s - 1) = 0$ so we can’t use Theorem 2.1 (b). In other words, the method of the proof Theorem 2.1 (b) can’t be used to find $\sum_{k=0}^{n} W_{2k}$ and $\sum_{k=0}^{n} W_{2k+1}$. Therefore we need another method to find them which is given in the following Theorem.

**Theorem 2.6.** If $r = 1, s = 2$ then for $n \geq 0$ we have the following formulas:

(a) $\sum_{k=0}^{n} W_k = \frac{1}{2}(W_{n+2} - W_1)$.
(b) $\sum_{k=0}^{n} W_{2k} = \frac{1}{2}(2W_{2n+2} - 2W_{2n+1} - W_0 + (-W_1 + 2W_0)n)$.
(c) $\sum_{k=0}^{n} W_{2k+1} = \frac{1}{8}(-W_{2n+2} + 10W_{2n+1} - 3W_1 + 2W_0 + (2W_1 - 4W_0)n)$.

Proof.

(a) Taking $r = 1, s = 2$ in Theorem 2.1 (a) we obtain (a).
(b) and (c) (b) and (c) can be proved by mathematical induction.

From the last Theorem we have the following Corollary which gives linear sum formulas of Jacobsthal numbers (take $W_n = J_n$ with $J_0 = 0, J_1 = 1$).

**Corollary 2.7.** For $n \geq 0$, Jacobsthal numbers have the following property:

(a) $\sum_{k=0}^{n} J_k = \frac{1}{2}(J_{n+2} - 1)$.
(b) $\sum_{k=0}^{n} J_{2k} = \frac{1}{2}(2J_{2n+2} - 2J_{2n+1} - n)$.
(c) $\sum_{k=0}^{n} J_{2k+1} = \frac{1}{4}(-J_{2n+2} + 10J_{2n+1} - 3 + 2n)$.

Taking $W_n = J_n$ with $j_0 = 2, j_1 = 1$ in the last Theorem, we have the following Corollary which presents linear sum formulas of Jacobsthal-Lucas numbers.

**Corollary 2.8.** For $n \geq 0$, Jacobsthal-Lucas numbers have the following property:

(a) $\sum_{k=0}^{n} J_k = \frac{1}{2}(2j_{n+2} - 1)$.
(b) $\sum_{k=0}^{n} J_{2k} = \frac{1}{4}(2j_{2n+2} - 2j_{2n+1} - 2 + 3n)$.
(c) $\sum_{k=0}^{n} J_{2k+1} = \frac{1}{8}(-j_{2n+2} + 10j_{2n+1} + 1 - 6n)$.
3 SUMMING FORMULAS OF GENERALIZED FIBONACCI NUMBERS WITH NEGATIVE SUBSCRIPTS

The following Theorem presents some linear summing formulas of generalized Fibonacci numbers with negative subscripts.

Theorem 3.1. For $n \geq 1$ we have the following formulas:

(a) (Sum of the generalized Fibonacci numbers with negative indices) If $r + s - 1 \neq 0$, then

$$\sum_{k=1}^{n} W_{-k} = \frac{-(r + s)W_{-n-1} - sW_{-n-2} + W_1 + (1 - r)W_0}{r + s - 1}.$$

(b) If $(r - s + 1)(r + s - 1) \neq 0$ then

$$\sum_{k=1}^{n} W_{-2k} = \frac{(s - 1)W_{-2n} - rsW_{-2n-1} + rW_1 + (1 - s - r^2)W_0}{(r - s + 1)(r + s - 1)}$$

and

$$\sum_{k=1}^{n} W_{-2k+1} = \frac{-rW_{-2n} + (s^2 - s)W_{-2n-1} + (1 - s)W_1 + rsW_0}{(r - s + 1)(r + s - 1)}.$$

(c) If $r \neq 0 \land s = 1$ then

$$\sum_{k=1}^{n} W_{-2k} = \frac{1}{r}(-W_{-2n-1} + W_1 - rW_0)$$

and

$$\sum_{k=1}^{n} W_{-2k+1} = \frac{1}{r}(-W_{-2n} + W_0).$$

Note that (c) is a special case of (b).

Proof.

(a) Using the recurrence relation

$$W_{-n+2} = rW_{-n+1} + sW_{-n}$$

i.e.

$$sW_{-n} = W_{-n+2} - rW_{-n+1}$$

or

$$W_{-n} = \frac{1}{s}W_{-n+2} - \frac{r}{s}W_{-n+1}$$

we obtain

$$sW_{-n} = W_{-n+2} - rW_{-n+1}$$

$$sW_{-n+1} = W_{-n+3} - rW_{-n+2}$$

$$sW_{-n+2} = W_{-n+4} - rW_{-n+3}$$

$$\vdots$$

$$sW_{-3} = W_{-1} - rW_{-2}$$

$$sW_{-2} = W_0 - rW_{-1}$$

$$sW_{-1} = W_1 - rW_0.$$
If we add the above equations by side by, we get
\[ \sum_{k=1}^{n} W_{-k} = \frac{(1 - r - s)W_{-n-1} - (r + s)W_{-n-2} - sW_{-n-3} + W_{1} + (1 - r)W_{0}}{r + s - 1}. \]

(b) and (c) Using the recurrence relation
\[ W_{-n+2} = rW_{-n+1} + sW_{-n} \]
i.e.
\[ rW_{-n+1} = W_{-n+2} - sW_{-n} \]
we obtain
\[
\begin{align*}
   rW_{-2n+1} &= W_{-2n+2} - sW_{-2n} \\
   rW_{-2n+3} &= W_{-2n+4} - sW_{-2n+2} \\
   rW_{-2n+5} &= W_{-2n+6} - sW_{-2n+4} \\
   &\vdots \\
   rW_{-5} &= W_{-4} - sW_{-6} \\
   rW_{-3} &= W_{-2} - sW_{-4} \\
   rW_{-1} &= W_{0} - sW_{-2}.
\end{align*}
\]
If we add the above equations by side by, we get
\[ r \sum_{k=1}^{n} W_{-2k+1} = (-W_{-2n} + W_{0} + \sum_{k=1}^{n} W_{-2k}) - s(\sum_{k=1}^{n} W_{-2k}). \quad (3.1) \]

Similarly, using the recurrence relation
\[ W_{-n+2} = rW_{-n+1} + sW_{-n} \]
i.e.
\[ rW_{-n} = W_{-n+1} - sW_{-n-1} \]
we obtain
\[
\begin{align*}
   rW_{-2n} &= W_{-2n+1} - sW_{-2n-1} \\
   rW_{-2n+2} &= W_{-2n+3} - sW_{-2n+1} \\
   rW_{-2n+4} &= W_{-2n+5} - sW_{-2n+3} \\
   &\vdots \\
   rW_{-4} &= W_{-3} - sW_{-5} \\
   rW_{-2} &= W_{-1} - sW_{-3}.
\end{align*}
\]
If we add the above equations by side by, we get
\[ r \sum_{k=1}^{n} W_{-2k} = (\sum_{k=1}^{n} W_{-2k+1}) - s(W_{-2n-1} - W_{-1} + \sum_{k=1}^{n} W_{-2k+1}). \]
Since
\[ W_{-1} = (-\frac{r}{s} \times W_{0} + \frac{1}{s} W_{1}) \]
it follows that
\[
\sum_{k=1}^{n} W_{-2k} = \left( \sum_{k=1}^{n} W_{-2k+1} \right) - s(W_{-2n-1} - (\frac{r}{s}) \times W_0 + \frac{1}{s} W_1) + \sum_{k=1}^{n} W_{-2k+1}.
\]

(3.2)

Then, solving system (3.1)-(3.2) the required result of (b) and (c) follow.

Taking \( r = s = 1 \) in Theorem 3.1 (a) and (b), we obtain the following Proposition.

**Proposition 3.1.** If \( r = s = 1 \) then for \( n \geq 1 \) we have the following formulas:

(a) \( \sum_{k=1}^{n} W_{-k} = -2W_{-n-1} - W_{-n-2} + W_1. \)

(b) \( \sum_{k=1}^{n} W_{-2k} = -W_{-2n-1} + W_1 - W_0. \)

(c) \( \sum_{k=1}^{n} W_{-2k+1} = -W_{-2n} + W_0. \)

From the above Proposition, we have the following Corollary which gives linear sum formulas of Fibonacci numbers (take \( W_n = F_n \) with \( F_0 = 0, F_1 = 1 \)).

**Corollary 3.2.** For \( n \geq 1 \), Fibonacci numbers have the following properties.

(a) \( \sum_{k=1}^{n} F_{-k} = -2F_{-n-1} - F_{-n-2} + 1. \)

(b) \( \sum_{k=1}^{n} F_{-2k} = -F_{-2n-1} + 1. \)

(c) \( \sum_{k=1}^{n} F_{-2k+1} = -F_{-2n}. \)

Taking \( W_n = L_n \) with \( L_0 = 2, L_1 = 1 \) in the last Proposition, we have the following Corollary which presents linear sum formulas of Lucas numbers.

**Corollary 3.3.** For \( n \geq 1 \), Lucas numbers have the following properties.

(a) \( \sum_{k=1}^{n} L_{-k} = -2L_{-n-1} - L_{-n-2} + 1. \)

(b) \( \sum_{k=1}^{n} L_{-2k} = -L_{-2n-1} - 1. \)

(c) \( \sum_{k=1}^{n} L_{-2k+1} = -L_{-2n} + 2. \)

Taking \( r = 2, s = 1 \) in Theorem 3.1 (a) and (b), we obtain the following Proposition.

**Proposition 3.2.** If \( r = 2, s = 1 \) then for \( n \geq 1 \) we have the following formulas:

(a) \( \sum_{k=1}^{n} W_{-k} = \frac{1}{2}(-3W_{-n-1} - W_{-n-2} + W_1 - W_0). \)

(b) \( \sum_{k=1}^{n} W_{-2k} = \frac{1}{2}(-W_{-2n-1} + W_1 - 2W_0). \)

(c) \( \sum_{k=1}^{n} W_{-2k+1} = \frac{1}{2}(-W_{-2n} + W_0). \)

From the last Proposition, we have the following Corollary which gives linear sum formulas of Pell numbers (take \( W_n = P_n \) with \( P_0 = 0, P_1 = 1 \)).

**Corollary 3.4.** For \( n \geq 1 \), Pell numbers have the following properties.

(a) \( \sum_{k=1}^{n} P_{-k} = \frac{1}{2}(-3P_{-n-1} - P_{-n-2} + 1). \)

(b) \( \sum_{k=1}^{n} P_{-2k} = \frac{1}{2}(-P_{-2n-1} + 1). \)

(c) \( \sum_{k=1}^{n} P_{-2k+1} = -\frac{1}{2}P_{-2n}. \)

Taking \( W_n = Q_n \) with \( Q_0 = 2, Q_1 = 2 \) in the last Proposition, we have the following Corollary which presents linear sum formulas of Pell-Lucas numbers.

**Corollary 3.5.** For \( n \geq 1 \), Pell-Lucas numbers have the following properties.
(a) \( \sum_{k=1}^{n} Q_{-k} = \frac{1}{2}(-3Q_{n-1} - Q_{n-2}). \)

(b) \( \sum_{k=1}^{n} Q_{-2k} = \frac{1}{2}(-Q_{-2n-1} - 2). \)

(c) \( \sum_{k=1}^{n} Q_{-2k+1} = \frac{1}{2}(-Q_{-2n+2}). \)

If \( r = 1, s = 2 \) then \( (r-s+1)(r+s-1) = 0 \) so we can’t use Theorem 3.1 (b). In other words, the method of the proof Theorem 3.1 (b) can’t be used to find \( \sum_{k=0}^{n} W_{2k} \) and \( \sum_{k=0}^{n} W_{2k+1} \). Therefore we need another method to find them which is given in the following Theorem.

**Theorem 3.6.** If \( r = 1, s = 2 \) then for \( n \geq 1 \) we have the following formulas:

(a) \[
\sum_{k=1}^{n} W_{-k} = \frac{1}{2}(-3W_{-n-1} - 2W_{-n-2} + W_1). \tag{3.3}
\]

(b) \[
\sum_{k=1}^{n} W_{-2k} = \frac{1}{3}(2W_{-2n-1} - 6W_{-2n-1} + (3W_1 - 5W_0) + (-W_1 + 2W_0)n). \tag{3.4}
\]

(c) \[
\sum_{k=1}^{n} W_{-2k+1} = \frac{1}{3}(2W_{-2n+1} - 6W_{-2n} + (-2W_1 + 6W_0) + (W_1 - 2W_0)n). \tag{3.5}
\]

**Proof.**

(a) Taking \( r = 1, s = 2 \) in Theorem 3.1 (a) we obtain (a).

(b) and (c) We prove (b). The proof will be by induction on \( n \). Before the proof, we recall some information on generalized Jacobsthal numbers. A generalized Jacobsthal sequence \( \{W_n\}_{n \geq 0} = \{W_n(W_0, W_1)\}_{n \geq 0} \) is defined by the second-order recurrence relations

\[
W_n = W_{n-1} + 2W_{n-2}; \quad W_0 = a, \quad W_1 = b, \quad (n \geq 2) \tag{3.6}
\]

with the initial values \( W_0, W_1 \) not all being zero. The sequence \( \{W_n\}_{n \geq 0} \) can be extended to negative subscripts by defining

\[
W_{-n} = \frac{1}{2}W_{-(n-1)} + \frac{1}{2}W_{-(n-2)}
\]

for \( n = 1, 2, 3, \ldots \). Therefore, recurrence (3.6) holds for all integer \( n \). The first few generalized Jacobsthal numbers with positive subscript and negative subscript are given in the following Table 1.

**Table 3. A few generalized Jacobsthal numbers**

<table>
<thead>
<tr>
<th>( n )</th>
<th>( W_n )</th>
<th>( W_{-n} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( W_0 )</td>
<td>( \ldots )</td>
</tr>
<tr>
<td>1</td>
<td>( W_1 )</td>
<td>( -\frac{1}{2}W_0 + \frac{1}{2}W_1 )</td>
</tr>
<tr>
<td>2</td>
<td>( 2W_0 + W_1 )</td>
<td>( \frac{3}{2}W_0 - \frac{1}{2}W_1 )</td>
</tr>
<tr>
<td>3</td>
<td>( 2W_0 + 3W_1 )</td>
<td>( -\frac{1}{2}W_0 + \frac{3}{2}W_1 )</td>
</tr>
<tr>
<td>4</td>
<td>( 6W_0 + 5W_1 )</td>
<td>( \frac{11}{6}W_0 - \frac{7}{6}W_1 )</td>
</tr>
<tr>
<td>5</td>
<td>( 10W_0 + 11W_1 )</td>
<td>( -\frac{23}{6}W_0 + \frac{11}{6}W_1 )</td>
</tr>
<tr>
<td>6</td>
<td>( 22W_0 + 21W_1 )</td>
<td>( \frac{43}{6}W_0 - \frac{21}{6}W_1 )</td>
</tr>
</tbody>
</table>
Binet formula of generalized Jacobsthal sequence can be calculated using its characteristic equation which is given as

\[ t^2 - t - 2 = 0. \]

The roots of characteristic equation are

\[ \alpha = 2, \quad \beta = -1 \]

and the roots satisfy the following

\[ \alpha + \beta = 1, \quad \alpha \beta = -2, \quad \alpha - \beta = 3. \]

Using these roots and the recurrence relation, Binet formula can be given as

\[ W_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta} = \frac{A2^n - B(-1)^n}{3} \tag{3.7} \]

where \( A = W_1 - W_0\beta = W_1 + W_0 \) and \( B = W_1 - W_0\alpha = W_1 - 2W_0 \).

We now prove (b) by induction on \( n \). If \( n = 1 \) we see that the sum formula reduces to the relation

\[ W_{-2} = \frac{1}{3}(2W_{-2} - 6W_{-3} + 2W_1 - 3W_0). \tag{3.8} \]

Since

\[ W_{-2} = \frac{3}{4}W_0 - \frac{1}{4}W_1, \quad W_{-3} = -\frac{5}{8}W_0 + \frac{3}{8}W_1, \]

(3.8) is true. Assume that (3.4) is true for \( n = m \). Then we get

\[ \sum_{k=1}^{m+1} W_{-2k} = \sum_{k=1}^{m} W_{-2k} + W_{-2(m+1)} = \frac{1}{3}(2W_{-2m} - 6W_{-2m-1} + (3W_1 - 5W_0) + (-W_1 + 2W_0)m) \]

\[ = \frac{1}{3}(2W_{-2m} - 6W_{-2m-1} + 3W_{-2m-2} + (3W_1 - 5W_0) + (-W_1 + 2W_0)m) \]

\[ = \frac{1}{3}(2W_{-2m} - 6W_{-2m-1} + 3W_{-2m-2} - (-W_1 + 2W_0) + (3W_1 - 5W_0) + (-W_1 + 2W_0)(m + 1)) \]

\[ = \frac{1}{3}(2W_{-2m} - 6W_{-2m-1} + 3W_{-2m-2} + (3W_1 - 5W_0) + (-W_1 + 2W_0)(m + 1)) \]

\[ = \frac{1}{3}(2W_{-2(m+1)} - 6W_{-(m+1)-1} + (3W_1 - 5W_0) + (-W_1 + 2W_0)(m + 1)) \]

where

\[ 2W_{-2m} - 6W_{-2m-1} + 3W_{-2m-2} - (-W_1 + 2W_0) = 2W_{-2m-2} - 6W_{-2m-3}. \tag{3.9} \]

(3.9) can be proved by using Binet formula of \( W_n \). Hence, the relation (3.4) holds also for \( n = m + 1 \).

Similarly, (c) can be proved by induction.

From the last Theorem, we have the following Corollary which gives sum formula of Jacobsthal numbers (take \( W_n = J_n \) with \( J_0 = 0, J_1 = 1 \)).

**Corollary 3.7.** For \( n \geq 1 \), Jacobsthal numbers have the following property:

(a) \[ \sum_{k=1}^{n} J_{-k} = \frac{1}{2}(-3J_{-n-1} - 2J_{-n-2} + 1). \]

(b) \[ \sum_{k=1}^{n} J_{-2k} = \frac{1}{3}(2J_{-2n} - 6J_{-2n-1} + 2n + 3 - n). \]
(c) \( \sum_{k=1}^{n} J_{-2k+1} = \frac{1}{2}(2J_{-2n+1} - 6J_{-2n} - 2 + n) \).

Taking \( W_n = j_n \) with \( j_0 = 2, j_1 = 1 \) in the last theorem, we have the following corollary which presents sum formulas of Jacobsthal-Lucas numbers.

**Corollary 3.8.** For \( n \geq 1 \), Jacobsthal-Lucas numbers have the following property:

(a) \( \sum_{k=1}^{n} j_{-k} = \frac{1}{2}(-3j_{-n-1} - 2j_{-n-2} + 1) \).
(b) \( \sum_{k=1}^{n} j_{-2k} = \frac{1}{2}(2j_{-2n} - 6j_{-2n-1} - 7 + 3n) \).
(c) \( \sum_{k=1}^{n} J_{-2k+1} = \frac{1}{2}(2j_{-2n+1} - 6j_{-2n} + 10 - 3n) \).

4 **SUMMING FORMULAS OF GAUSSIAN GENERALIZED FIBONACCI NUMBERS WITH POSITIVE SUBSCRIPTS**

The following theorem presents some linear summing formulas of Gaussian generalized Fibonacci numbers with positive subscripts.

**Theorem 4.1.** For \( n \geq 0 \) we have the following formulas:

(a) (Sum of the generalized Gaussian Fibonacci numbers) If \( r + s - 1 \neq 0 \), then

\[
\sum_{k=0}^{n} GW_k = GW_{n+2} + (1-r)GW_{n+1} - GW_1 + (r-1)GW_0. 
\]

(b) If \( (r-s+1)(r+s-1) \neq 0 \) then

\[
\sum_{k=0}^{n} GW_{2k} = \frac{(1-s)GW_{2n+2} + rsGW_{2n+1} + (s-1)GW_2 - rsGW_1 + (r^2-s^2+2s-1)GW_0}{(r-s+1)(r+s-1)}.
\]

and

\[
\sum_{k=0}^{n} GW_{2k+1} = \frac{rGW_{2n+2} + (s-s^2)GW_{2n+1} - rGW_2 + (r^2+s-1)GW_1}{(r-s+1)(r+s-1)}.
\]

(c) If \( r \neq 0 \land s = 1 \) then

\[
\sum_{k=0}^{n} GW_{2k} = \frac{GW_{2n+1} - GW_1 + rGW_0}{r}
\]

and

\[
\sum_{k=0}^{n} GW_{2k+1} = \frac{GW_{2n+2} - GW_2 + rGW_1}{r}.
\]

Note that (c) is a special case of (b).

**Proof.** The proof can be given exactly as in the proof of Theorem 2.1.

Taking \( r = s = 1 \) in Theorem 4.1 (a) and (b), we obtain the following proposition.

**Proposition 4.1.** If \( r = s = 1 \) then for \( n \geq 0 \) we have the following formulas:

(a) \( \sum_{k=0}^{n} GW_k = GW_{n+2} - GW_1 \).
(b) \( \sum_{k=0}^{n} GW_{2k} = GW_{2n+1} - GW_1 + GW_0 \).
(c) \( \sum_{k=0}^{n} GW_{2k+1} = GW_{2n+2} - GW_2 + GW_1 \).
From the above Proposition, we have the following Corollary which gives linear sum formulas of Gaussian Fibonacci numbers (take $GW_n = GF_n$ with $GF_0 = i, GF_1 = 1$).

**Corollary 4.2.** For $n \geq 0$, Gaussian Fibonacci numbers have the following properties:

(a) $\sum_{k=0}^{n} k = GF_{n+2} - 1$.
(b) $\sum_{k=0}^{n} GF_{2k} = GF_{2n+1} - (1 - i)$.
(c) $\sum_{k=0}^{n} GF_{2k+1} = GF_{2n+2} - i$.

Taking $GW_n = GL_n$ with $GL_0 = 2 - i, GL_1 = 1 + 2i$ in the last Proposition, we have the following Corollary which presents linear sum formulas of Gaussian Lucas numbers.

**Corollary 4.3.** For $n \geq 0$, Gaussian Lucas numbers have the following properties:

(a) $\sum_{k=0}^{n} GL_k = GL_{n+2} - (1 + 2i)$.
(b) $\sum_{k=0}^{n} GL_{2k} = GL_{2n+1} + (1 - 3i)$.
(c) $\sum_{k=0}^{n} GL_{2k+1} = GL_{2n+2} + (-2 + i)$.

Taking $r = 2, s = 1$ in Theorem 4.1 (a) and (b), we obtain the following Proposition.

**Proposition 4.2.** If $r = 2, s = t = 1$ then for $n \geq 0$ we have the following formulas:

(a) $\sum_{k=0}^{n} GW_k = \frac{1}{2}(GW_{n+2} - GW_{n+1} - GW_1 + GW_0)$.
(b) $\sum_{k=0}^{n} GW_{2k} = \frac{1}{2}(GW_{2n+1} - GW_1 + 2GW_0)$.
(c) $\sum_{k=0}^{n} GW_{2k+1} = \frac{1}{2}(GW_{2n+2} - GW_2 + 2GW_1)$.

From the last Proposition, we have the following Corollary which gives linear sum formulas of Gaussian Pell numbers (take $GW_n = GP_n$ with $GP_0 = i, GP_1 = 1$).

**Corollary 4.4.** For $n \geq 0$, Gaussian Pell numbers have the following properties:

(a) $\sum_{k=0}^{n} GP_k = \frac{1}{2}(GP_{n+2} - GP_{n+1} - (1 - i))$.
(b) $\sum_{k=0}^{n} GP_{2k} = \frac{1}{2}(GP_{2n+1} - (1 - 2i))$.
(c) $\sum_{k=0}^{n} GP_{2k+1} = \frac{1}{2}(GP_{2n+2} - i)$.

Taking $GW_n = GQ_i$ with $GQ_0 = 2 - 2i, GQ_1 = 2 + 2i$ in the last Proposition, we have the following Corollary which presents linear sum formulas of Gaussian Pell-Lucas numbers.

**Corollary 4.5.** For $n \geq 0$, Gaussian Pell-Lucas numbers have the following properties:

(a) $\sum_{k=0}^{n} GQ_k = \frac{1}{2}(GQ_{n+2} - GQ_{n+1} - 4i)$.
(b) $\sum_{k=0}^{n} GQ_{2k} = \frac{1}{2}(GQ_{2n+1} + (2 - 6i))$.
(c) $\sum_{k=0}^{n} GQ_{2k+1} = \frac{1}{2}(GQ_{2n+2} - (2 - 2i))$.

If $r = 1, s = 2$ then $(r - s + 1) (r + s - 1) = 0$ so we can’t use Theorem 4.1 (b). In other words, the method of the proof Theorem 4.1 (b) can’t be used to find $\sum_{k=0}^{n} GW_{2k}$ and $\sum_{k=0}^{n} GW_{2k+1}$. Therefore we need another method to find them which is given in the following Theorem.

**Theorem 4.6.** If $r = 1, s = 2$ then for $n \geq 0$ we have the following formulas:

(a) $\sum_{k=0}^{n} GW_k = \frac{1}{2}(GW_{n+2} - GW_1)$.
(b) $\sum_{k=0}^{n} GW_{2k} = \frac{1}{2}(2GW_{n+2} - 2GW_{2n+1} - GW_0 + (-GW_1 + 2GW_0)n)$.
(c) $\sum_{k=0}^{n} GW_{2k+1} = \frac{1}{8}(-GW_{2n+2} + 10GW_{2n+1} - 3GW_1 + 2GW_0 + (2GW_1 - 4GW_0)n)$. 
Proof.

(a) Taking \( r = 1, s = 2 \) in Theorem 4.1 (a) we obtain (a).

(b) and (c) (b) and (c) can be proved by mathematical induction.

From the last Theorem we have the following Corollary which gives linear sum formulas of Gaussian Jacobsthal numbers (take \( GW_n = GJ_n \) with \( GJ_0 = \frac{1}{2}i, GJ_1 = 1 \)).

**Corollary 4.7.** For \( n \geq 0 \), Gaussian Jacobsthal numbers have the following property:

(a) \( \sum_{k=0}^{n} GJ_k = \frac{1}{2}(GJ_{n+2} - 1). \)

(b) \( \sum_{k=0}^{n} GJ_{2k} = \frac{1}{4}(2GJ_{2n+2} - 2GJ_{2n+1} - \frac{1}{2}i - (1 - i)n). \)

(c) \( \sum_{k=0}^{n} GJ_{2k+1} = \frac{1}{4}(-GJ_{2n+2} + 10GJ_{2n+1} + (-3 + i) + (2 - 2i)n). \)

Taking \( GW_n = GJ_n \) with \( GJ_0 = \frac{1}{2}i, GJ_1 = 1 + 2i \) in the last Theorem, we have the following Corollary which presents linear sum formulas of Gaussian Jacobsthal-Lucas numbers.

**Corollary 4.8.** For \( n \geq 0 \), Gaussian Jacobsthal-Lucas numbers have the following property:

(a) \( \sum_{k=0}^{n} GJ_k = \frac{1}{2}(GJ_{n+2} - (1 + 2i)). \)

(b) \( \sum_{k=0}^{n} GJ_{2k} = \frac{1}{4}(2GJ_{2n+2} - 2GJ_{2n+1} - 2 - \frac{1}{2}i + (3 - 3i)n). \)

(c) \( \sum_{k=0}^{n} GJ_{2k+1} = \frac{1}{4}(-GJ_{2n+2} + 10GJ_{2n+1} + (1 - 7i) + (-6 + 6i)n). \)

5 SUMMING FORMULAS OF GAUSSIAN GENERALIZED FIBONACCI NUMBERS WITH NEGATIVE SUBSCRIPTS

The following Theorem presents some linear sum formulas of Gaussian generalized Fibonacci numbers with negative subscripts.

**Theorem 5.1.** For \( n \geq 1 \) we have the following formulas:

(a) (Sum of the generalized Gaussian Fibonacci numbers with negative indices) If \( r + s - 1 \neq 0 \), then

\[
\sum_{k=0}^{n} GW_{-k} = \frac{-(r + s)GW_{-n-1} - sGW_{-n-2} + GW_1 + (1 - r)GW_0}{r + s - 1}.
\]

(b) If \( (r - s + 1)(r + s - 1) \neq 0 \) then

\[
\sum_{k=1}^{n} GW_{-2k} = \frac{(s - 1)GW_{-2n} - rsGW_{-2n-1} + rGW_1 + (1 - s - r^2)GW_0}{(r - s + 1)(r + s - 1)}
\]

and

\[
\sum_{k=1}^{n} GW_{-2k+1} = \frac{-rGW_{-2n} + (s^2 - s)GW_{-2n-1} + (1 - s)GW_1 + rsGW_0}{(r - s + 1)(r + s - 1)}.
\]

(c) If \( r \neq 0 \land s = 1 \) then

\[
\sum_{k=1}^{n} GW_{-2k} = \frac{1}{r}(-GW_{-2n-1} + GW_1 - rGW_0)
\]

and

\[
\sum_{k=1}^{n} GW_{-2k+1} = \frac{1}{r}(-GW_{-2n} + GW_0).
\]

Note that (c) is a special case of (b).
For (a) and (b), we obtain the following Proposition.

Proposition 5.1. If \( r = s = 1 \) then for \( n \geq 1 \) we have the following formulas:

(a) \[ \sum_{k=1}^{n} GW_{-k} = -2GW_{-n-1} - GW_{-n-2} + GW_1. \]
(b) \[ \sum_{k=1}^{n} GW_{-2k} = -GW_{-2n-1} + GW_1 - GW_0. \]
(c) \[ \sum_{k=1}^{n} GW_{-2k+1} = -GW_{-2n} + GW_0. \]

From the above Proposition, we have the following Corollary which gives linear sum formulas of Gaussian Fibonacci numbers. (take \( GW_n = GF_n \) with \( GF_0 = i, GF_1 = 1 \)).

Corollary 5.2. For \( n \geq 1 \), Gaussian Fibonacci numbers have the following properties.

(a) \[ \sum_{k=1}^{n} GF_{-k} = -2GF_{-n-1} - GF_{-n-2} + 1. \]
(b) \[ \sum_{k=1}^{n} GF_{-2k} = -GF_{-2n-1} + 1 - i. \]
(c) \[ \sum_{k=1}^{n} GF_{-2k+1} = -GF_{-2n} + i. \]

Taking \( GW_n = GL_n \) with \( GL_0 = 2 - i, GL_1 = 1 + 2i \) in the last Proposition, we have the following Corollary which presents linear sum formulas of Gaussian Lucas numbers.

Corollary 5.3. For \( n \geq 1 \), Gaussian Lucas numbers have the following properties.

(a) \[ \sum_{k=1}^{n} GL_{-k} = -2GL_{-n-1} - GL_{-n-2} + (1 + 2i). \]
(b) \[ \sum_{k=1}^{n} GL_{-2k} = -GL_{-2n-1} + (-1 + 3i). \]
(c) \[ \sum_{k=1}^{n} GL_{-2k+1} = -GL_{-2n} + (2 - i). \]

Taking \( r = 2, s = 1 \) in Theorem 5.1 (a) and (b), we obtain the following Proposition.

Proposition 5.2. If \( r = 2, s = 1 \) then for \( n \geq 1 \) we have the following formulas:

(a) \[ \sum_{k=1}^{n} GW_{-k} = \frac{1}{2}(-3GW_{-n-1} - GW_{-n-2} + GW_1 - GW_0). \]
(b) \[ \sum_{k=1}^{n} GW_{-2k} = \frac{1}{2}(-GW_{-2n-1} + GW_1 - 2GW_0). \]
(c) \[ \sum_{k=1}^{n} GW_{-2k+1} = \frac{1}{2}(-GW_{-2n} + GW_0). \]

From the last Proposition, we have the following Corollary which gives linear sum formulas of Gaussian Pell numbers. (take \( GW_n = GP_n \) with \( GP_0 = i, GP_1 = 1 \)).

Corollary 5.4. For \( n \geq 1 \), Gaussian Pell numbers have the following properties.

(a) \[ \sum_{k=1}^{n} GP_{-k} = \frac{1}{2}(-3GP_{-n-1} - GP_{-n-2} + 1). \]
(b) \[ \sum_{k=1}^{n} GP_{-2k} = \frac{1}{2}(-GP_{-2n-1} + 1). \]
(c) \[ \sum_{k=1}^{n} GP_{-2k+1} = -\frac{1}{2}GP_{-2n}. \]

Taking \( GW_n = GQ_n \) with \( GQ_0 = 2 - 2i, GQ_1 = 2 + 2i \) in the last Proposition, we have the following Corollary which presents linear sum formulas of Gaussian Pell-Lucas numbers.

Corollary 5.5. For \( n \geq 1 \), Gaussian Pell-Lucas numbers have the following properties.

(a) \[ \sum_{k=1}^{n} GQ_{-k} = \frac{1}{2}(-3GQ_{-n-1} - GQ_{-n-2} + 4i). \]
(b) \[ \sum_{k=1}^{n} GQ_{-2k} = \frac{1}{2}(-GQ_{-2n-1} - (2 - 6i)). \]
(c) \[ \sum_{k=1}^{n} GQ_{-2k+1} = \frac{1}{2}(-GQ_{-2n} + (2 - 2i)). \]
If \( r = 1, s = 2 \) then \((r-s+1)(r+s-1) = 0\) so we can’t use Theorem 5.1 (b). In other words, the method of the proof Theorem 5.1 (b) can’t be used to find \(\sum_{k=0}^{n} GW_{2k}\) and \(\sum_{k=0}^{n} GW_{2k+1}\). Therefore we need another method to find them which is given in the following Theorem.

**Theorem 5.6.** If \( r = 1, s = 2 \) then for \( n \geq 1 \) we have the following formulas:

(a) \(\sum_{k=1}^{n} GW_{-k} = \frac{1}{2}(-3GW_{-n-1} - 2GW_{-n-2} + GW_{1})\).

(b) \(\sum_{k=1}^{n} GW_{-2k} = \frac{1}{2}(2GW_{-2n} - 6GW_{-2n-1} + (3GW_{1} - 5GW_{0}) + (-GW_{1} + GW_{0})n)\).

(c) \(\sum_{k=1}^{n} GW_{-2k+1} = \frac{1}{2}(2GW_{-2n+1} - 6GW_{-2n} + (-2GW_{1} + 6GW_{0}) + (GW_{1} - 2GW_{0})n)\).

**Proof.**

(a) Taking \( r = 1, s = 2 \) in Theorem 5.1 (a) we obtain (a).

(b) and (c) the proof of (b) and (c) can be given exactly as in the proof of Theorem 3.6 (b) and (c) using mathematical induction.

From the last Theorem, we have the following Corollary which gives sum formula of Gaussian Jacobsthal numbers (take \( GW_{n} = GJ_{n} \) with \( GJ_{0} = \frac{1}{2}i, GJ_{1} = 1 \)).

**Corollary 5.7.** For \( n \geq 1 \), Gaussian Jacobsthal numbers have the following property:

(a) \(\sum_{k=1}^{n} GJ_{-k} = \frac{1}{2}(-3GJ_{-n-1} - 2GJ_{-n-2} + 1)\).

(b) \(\sum_{k=1}^{n} GJ_{-2k} = \frac{1}{2}(2GJ_{-2n} - 6GJ_{-2n-1} + (3 - \frac{7}{2}i) + (-1 + i)n)\).

(c) \(\sum_{k=1}^{n} GJ_{-2k+1} = \frac{1}{2}(2GJ_{-2n+1} - 6GJ_{-2n} + (-2 + 3i) + (1 - i)n)\).

Taking \( GW_{n} = GJ_{n} \) with \( GJ_{0} = 2 - \frac{1}{2}i, GJ_{1} = 1 + 2i \) in the last Theorem, we have the following Corollary which presents sum formulas of Gaussian Jacobsthal-Lucas numbers.

**Corollary 5.8.** For \( n \geq 1 \), Gaussian Jacobsthal-Lucas numbers have the following property:

(a) \(\sum_{k=1}^{n} GJ_{-k} = \frac{1}{2}(-3GJ_{-n-1} - 2GJ_{-n-2} + (1 + 2i))\).

(b) \(\sum_{k=1}^{n} GJ_{-2k} = \frac{1}{2}(2GJ_{-2n} - 6GJ_{-2n-1} + (-7 + \frac{15}{2}i) + (3 - 3i)n)\).

(c) \(\sum_{k=1}^{n} GJ_{-2k+1} = \frac{1}{2}(2GJ_{-2n+1} - 6GJ_{-2n} + (10 - 7i) + (-3 + 3i)n)\).

**6 CONCLUSION**

In this work, a number of linear and a few non-linear sum identities were discovered and proved. The method used in this paper can be used for the other linear recurrence sequences, too. We have written linear sum identities in terms of the generalized Fibonacci sequence, and then we have presented the formulas as special cases the corresponding identity for the special cases of the generalized Fibonacci Gaussian generalized Fibonacci sequences such as Fibonacci-Lucas sequence and Gaussian Fibonacci-Lucas sequence.

All the listed identities may be proved by induction, but that method of proof gives no clue about their discovery. We give the proofs to indicate how these identities, in general, were discovered. Recently, there have been so many studies of the sequences of numbers in the literature and the sequences of numbers were widely used in many research areas, such as architecture, nature, art, physics and engineering. See, for example, the articles [15], [16].

Our next publication will be about summation formulas for generalized Tribonacci and Gaussian generalized Tribonacci numbers using similar methods of this paper. Also we plan to investigate summation formulas for generalized Tetranacci, Gaussian generalized Tetranacci numbers and for generalized Pentanacci, Gaussian generalized Pentanacci numbers.
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COMPETING INTERESTS

Author has declared that no competing interests exist.

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